

Vortex shedding and drag in dilute Bose-Einstein condensates

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Abstract.

Above a critical velocity, the dominant mechanism of energy transfer between a moving object and a dilute Bose-Einstein condensate is vortex formation. In this paper, we discuss the critical velocity for vortex formation and the link between vortex shedding and drag in both homogeneous and inhomogeneous condensates. We find that at supersonic velocities sound radiation also contributes significantly to the drag force.

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1. Introduction

One enticing consequence of the discovery of Bose-Einstein condensation (BEC) in dilute alkali vapours [1] is the potential for refining our understanding of quantum fluids. In particular, the dilute Bose gas provides a near-ideal testing ground for elucidating the role of vortices in the onset of dissipation in superfluids. Recent experiments have demonstrated the formation of quantized vortices by rotational excitation of one [2] and two-component condensates [3], in analogy with the famous ‘rotating bucket’ experiments in liquid helium [4]. In addition, by moving a far-off resonant laser beam through a condensate, Raman *et al.* [5] measure a heating rate which suggests a critical velocity for dissipation characteristic of vortex shedding.

The attractive feature of experiments on dilute Bose gases is that the weakly-interacting limit permits quantitative comparison between theory and experiment. The theoretical description is based on the Gross-Pitaevskii equation, a form of non-linear Schrödinger equation (NLSE) [6]. In the NLSE model, the critical velocity for vortex shedding by a cylindrical object was found to be, $v_c \sim 0.4c$, where c is the speed of sound [7]. In a trapped (inhomogeneous) condensate the critical velocity is lower due to the reduction in density, and hence sound speed, towards the edge of the condensate [5, 8].

In this paper, we study vortex shedding due to the motion of an object through homogeneous and trapped Bose-Einstein condensates using the NLSE model. The hydrodynamical properties of a NLSE fluid are reviewed in Sec. 2. In Sec. 3 and 4 we discuss the critical velocity for vortex nucleation and the link between vortex

formation and drag in homogeneous condensates. In Sec. 5 we consider the properties of trapped condensates and highlight the differences from the homogeneous case.

2. Quantum fluid mechanics

At low temperatures and low densities, atoms interact by elastic s -wave scattering, and collisions can be parameterised by a single variable, the scattering length, a . For atoms of mass m , the wavefunction of the condensate, $\psi(\mathbf{r}, t)$, is given by the solution of the time-dependent Schrödinger equation:

$$i\hbar\partial_t\psi(\mathbf{r}, t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r}, t) + g|\psi(\mathbf{r}, t)|^2 \right] \psi(\mathbf{r}, t) , \quad (1)$$

where the wavefunction is normalised to the number of atoms, N , the coefficient of the non-linear term, $g = 4\pi\hbar^2 a/m$, describes the interactions within the fluid, and $V(\mathbf{r}, t)$ represents external potentials arising from the trap and any moving obstacle.

2.1. Fluid equations

The link between the nonlinear Schrödinger equation (1) and the equivalent equations of fluid mechanics is well known [4, 9]. However some points concerning condensate flow near penetrable objects are less familiar. In this section we gather some of the key concepts and equations that figure in the discussion of our simulations.

Classical (isentropic) fluid mechanics is based on two coupled differential equations: one describing the transport of mass, the other the transport of momentum [10]. The relevant quantum variables can be constructed from the wavefunction: the mass density ρ and momentum current density \mathbf{J} are defined as,

$$\rho \equiv m\psi^*\psi \quad \text{and} \quad J_k \equiv (\hbar/2i)(\psi^*\partial_k\psi - \psi\partial_k\psi^*) , \quad (2)$$

where the index k denotes the vector component. The fluid velocity is defined by $v_k \equiv J_k/\rho$, or equivalently in terms of the phase, ϕ , of the wavefunction, $v_k \equiv (\hbar/m)\partial_k\phi$. Clearly, the velocity field is a potential flow, however it is also compressible and furthermore can support circulation (vorticity) as will be seen.

The conservation of mass (probability), i.e., the continuity equation, follows from the definition of ρ and equation (1)

$$\partial_t\rho + \partial_k J_k = 0 . \quad (3)$$

The conservation of momentum equation may be found by considering the rate of change of the momentum current density,

$$\partial_t J_k + \partial_j T_{jk} + \rho\partial_k(V/m) = 0 , \quad (4)$$

where the momentum flux density tensor takes the form [7, 11],

$$T_{jk} = \frac{\hbar^2}{4m}(\partial_j\psi^*\partial_k\psi - \psi^*\partial_j\partial_k\psi + \text{c.c.}) + \frac{g}{2}\delta_{jk}|\psi|^4 . \quad (5)$$

This can be rewritten as,

$$T_{jk} = \rho v_j v_k - \sigma_{jk} , \quad (6)$$

where the stress tensor σ_{jk} is given by,

$$\sigma_{jk} = -\frac{1}{2}\delta_{jk}g(\rho/m)^2 + (\hbar/2m)^2\rho\partial_j\partial_k\ln\rho . \quad (7)$$

2.2. Pressure, sound and drag

The form of equations (3), (4), and (6) is identical to those for classical fluid flow [10], the difference emerges from the nature of the stress, equation (7). A classical ideal fluid is characterised by $\sigma_{jk} = 0$, for all j, k . In a viscous fluid, the shear stress ($\sigma_{jk}, j \neq k$) is produced by velocity gradients between neighbouring streams such that, $\sigma_{jk} = \eta(\partial_j v_k + \partial_k v_j)$, where η is the coefficient of viscosity. This creates a frictional force which gives rise to energy loss. In a pure dilute Bose-Einstein condensate there is no frictional viscosity, but a shear stress arises from density gradients, the second term in equation (7). This property gives rise to the possibility of vortex formation and drag without viscosity.

The pressure (normal stress, $-\sigma_{jk}, j = k$) within the quantum fluid takes the simple form,

$$p = \frac{1}{2}g(\rho/m)^2 - (\hbar/2m)^2 \rho \nabla^2 \ln \rho . \quad (8)$$

The second term, called the quantum pressure, is weak in homogeneous regions of the fluid, that is, far from obstacles or boundaries, vortex lines or shocks. The essential difference between interacting and noninteracting (ideal) fluids is the existence of interaction pressure which supports sound propagation. In the bulk of the fluid, where the quantum pressure is negligible, the speed of sound [9, 10],

$$c = \sqrt{\partial p / \partial \rho} = (g\rho/m^2)^{\frac{1}{2}} . \quad (9)$$

The force on an obstacle moving through a condensate can be calculated from the rate of momentum transfer to the fluid. By integrating Eq. (4), one finds that the k -th component of the force,

$$F_k = \partial_t \int_{\Omega} d\Omega J_k = - \int_S dS n_j T_{jk} - \int_{\Omega} d\Omega \rho \partial_k (V/m) , \quad (10)$$

where S is the surface of the object or control surface within the fluid [11], Ω is the volume enclosed by S , n_j is the j -component of the normal vector to S , and dS is a surface element. The second term on the right-hand side can be likened to the buoyancy of the fluid. In the case of homogeneous flow past an impenetrable object (Sec. 3 and 4), the wavefunction vanishes on the object surface and the potential is uniform elsewhere, therefore, only the first term contributes. Conversely, for a penetrable object in a trapped condensate (Sec. 5), Ω may be chosen to encompass the entire fluid, and the first term is negligible compared to the second.

2.3. Quantisation of circulation

The quantum Euler equation follows from combining the equations describing the conservation of mass and momentum, (3) and (4), along with the identity,

$$\rho^{-1} \partial_j [\rho \partial_j \partial_k \ln \rho] = 2 \partial_k [\rho^{-\frac{1}{2}} \partial_j \partial_j \rho^{\frac{1}{2}}] , \quad (11)$$

allowing the momentum equation to be written as,

$$\partial_t v_k + v_j \partial_j v_k + \partial_k [g\rho/m^2 - (\hbar^2/2m)\rho^{-\frac{1}{2}} \partial_j \partial_j \rho^{\frac{1}{2}} + V/m] = 0 . \quad (12)$$

The conservation of energy (Bernoulli equation) then follows as an integral of Euler's equation, or more directly from the real part of equation (1):

$$\hbar \partial_t \phi + \frac{1}{2} m v^2 + g\rho/m - (\hbar^2/2m)\rho^{-\frac{1}{2}} \nabla^2 \rho^{\frac{1}{2}} + V = 0 . \quad (13)$$

Perhaps the most significant quantum effect on the mechanics of the fluid is the quantisation of angular momentum. The circulation is given by,

$$\Gamma = \oint d\mathbf{r} \cdot \mathbf{v} = (\hbar/m)2\pi s \quad s = 0, 1, 2, \dots \quad (14)$$

where the closed contour joins fluid particles. The conservation of angular momentum (Kelvin's theorem), follows from Euler's equation (12) and states that the circulation around a closed 'fluid' contour does not change in time. This means that within the fluid, vortex lines must be created in pairs which emerge from a point. The exception is at boundaries, where the wavefunction is clamped to zero and no closed fluid loop can be drawn, e.g., at the surface of an impenetrable object [10] or from the edge of a trapped condensate [13].

2.4. Units

For a homogeneous fluid flow, where the external potential is due to the obstacle only, it is convenient to rescale length and velocity in terms of the healing length $\xi = \hbar/\sqrt{mn_0g}$ and the asymptotic speed of sound $c = \sqrt{n_0g/m}$, respectively. In this case, equation (1) becomes

$$i\partial_t \tilde{\psi}(\mathbf{r}, t) = \left[-\frac{1}{2}\nabla^2 + V(\mathbf{r}') + |\tilde{\psi}(\mathbf{r}, t)|^2 \right] \tilde{\psi}(\mathbf{r}, t), \quad (15)$$

where $\tilde{\psi} = \psi/\sqrt{n_0}$ and n_0 is the number density far from the object. The force per unit length is measured in units of $\hbar\sqrt{n_0^3g/m}$. Unless otherwise stated we use these units throughout. For steady flow, in which \mathbf{v} and n are independent of time and $\phi = -\mu t$, where μ is the chemical potential, the Bernoulli equation takes the form

$$n - \frac{1}{2}(\sqrt{n})^{-1}\nabla^2\sqrt{n} + V + \frac{1}{2}v^2 = \text{constant}. \quad (16)$$

3. The critical velocity

The critical velocity for the breakdown of superfluidity is often expressed in terms of the Landau condition [9], $v_c = (\epsilon/p)_{\min}$, where ϵ and p are the energy and momentum of elementary excitations in the fluid, and v_c is the flow velocity in the fluid bulk. In the dilute Bose gas, the long wavelength elementary excitations are sound waves and the Landau criterion predicts that $v_c = c$. However, for flow past an object, the local velocity near the obstacle, v , can become supersonic even when flow velocity, U , is subsonic. Consequently, the critical flow velocity, v_c , where laminar flow becomes unstable occurs at a fraction of the sound speed. An estimate of v_c may be found following the argument suggested by Frisch *et al.* [7]. For an incompressible flow past a solid object, Bernoulli's equation (16) (neglecting the quantum pressure) has the simple form,

$$n(v) + \frac{1}{2}v^2 = 1 + \frac{1}{2}U^2, \quad (17)$$

where U is the background flow velocity. The maximum velocity which occurs at the equator of the object is $v = \frac{3}{2}U$ for a sphere (or $v = 2U$ for a cylinder), therefore, $\frac{1}{2}(\frac{9}{4} - 1)U^2 = 1 - n(v)$. The critical velocity is reached when the maximum speed, $v = \frac{3}{2}U$, is equal to the 'bulk' sound speed, $c = \sqrt{n(v)}$, which gives $v_c = U = \sqrt{8/23} \approx 0.59$ (or $\sqrt{2/11} \approx 0.44$ for a cylinder). However, for a compressible fluid, the equatorial velocity is slightly larger due to pressure effects. The first-order correction gives $v = \frac{3}{2}U + \frac{1}{2}U^3$ which reduces the critical velocity to $v_c \approx 0.53$.

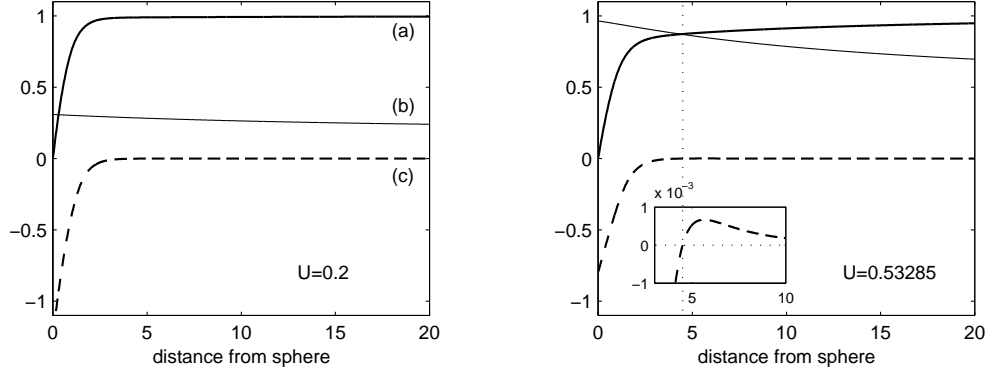


Figure 1. Laminar flow past a sphere with radius $R = 50$ for two flow velocities $U = 0.2$ (left) and $U = 0.53285$ (right). The three curves show, (a) the ‘bulk’ sound speed $c = \sqrt{n} = |\psi|$, (b) the velocity, v , and (c) the quantum pressure, $\nabla^2 \psi / 2\psi$, as a function of position. Note that, at the critical velocity $u = 0.53285$ (right), the fluid velocity is equal to the sound speed where the quantum pressure is exactly zero.

The exact value of v_c may be found by solving the uniform flow equation [14, 15],

$$i\partial_t \tilde{\psi}'(\mathbf{r}', t) = \left[-\frac{1}{2} \nabla'^2 + V(\mathbf{r}') + |\tilde{\psi}'(\mathbf{r}', t)|^2 + i\mathbf{v} \cdot \nabla' \right] \tilde{\psi}'(\mathbf{r}', t), \quad (18)$$

where $\tilde{\psi}'(\mathbf{r}', t) = \tilde{\psi}(\mathbf{r}, t)$ is the wavefunction in the fluid rest frame written in terms of the object frame coordinates, $\mathbf{r}' = \mathbf{r} - \mathbf{v}t$. Stationary solutions of the form, $\tilde{\psi}'(\mathbf{r}', t) = \phi(\mathbf{r}')e^{-i\mu t}$ are found to exist only for $v \leq v_c$, where v_c is the critical velocity for vortex formation.

To illustrate the behaviour of the exact solutions near the critical velocity, we solve equation (18) in 3D for an impenetrable sphere with radius $R = 50$. The wavefunction, velocity and quantum pressure term near the object are shown in Fig. 2. Note that these parameters are related via the Bernoulli equation (16). The intersection between the velocity v and wavefunction amplitude, $|\psi|$, curves defines the position where the velocity is equal to the ‘bulk’ sound speed (9). Note that close to the object the effective sound speed is increased due to the quantum pressure term, (8), therefore even though the density is low the flow is not ‘supersonic’. The critical velocity is reached when flow velocity exceeds the speed of sound in the bulk of the fluid, i.e., when the intersection between the velocity and wavefunction curves moves into the region where the quantum pressure term is zero, see Fig. 1(right).

The complication for trapped condensates is that the speed of sound and hence the critical velocity depend upon position. In addition, the object potential is typically penetrable and non-uniform. We return to these topics in Sec. 5.

4. Vortex shedding, drag, and dissipation

For flow faster than the critical velocity, $v > v_c$, vortices are emitted approximately periodically. A typical vortex stream pattern for flow past an impenetrable cylinder is shown in Fig. 2. The background flow is from right to left and the vortex - anti-vortex pairs produce a flow pattern which opposes the background. Consequently, the vortex

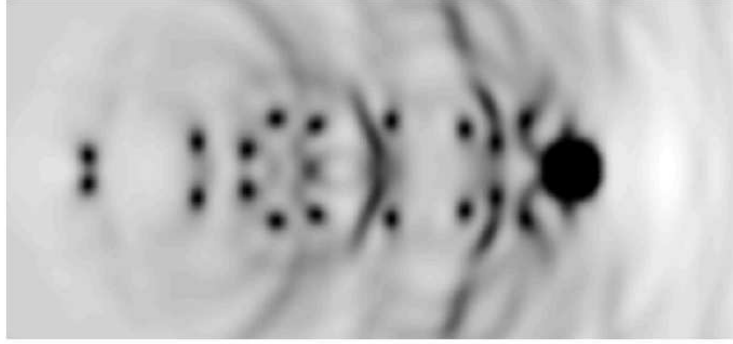


Figure 2. A shaded contour plot showing the condensate density for homogeneous flow past a cylinder with radius $R = 3$. The dark regions indicate lower density, i.e., the position of the object and vortex lines.

trail separates the main flow from an almost stationary wake. The momentum loss from the fluid is transferred to the object creating a drag force. The contribution of vortex shedding to the drag force can be estimated by considering the momentum transfer due to vortex emission, i.e.,

$$\mathbf{F}_v = f_v \mathbf{p}_v, \quad (19)$$

where f_v is the vortex shedding frequency and \mathbf{p}_v is the momentum of a vortex pair as it is created at the equatorial plane. Small fluctuations in the vortex shedding frequency occur because as the vortices move downstream they interact with each other creating fluctuations in the flow pattern around the object. The drag force is taken to be the time-average over many vortex emission cycles.

In addition to vortex shedding, drag may also arise due to sound waves. The time-average drag, evaluated using equation (10), as a function of velocity for an impenetrable cylinder is shown in Fig. 3. The drag is zero up to the critical velocity, then increases approximately quadratically with v [7, 11]. Also shown is the contribution to the drag force from vortex shedding alone, i.e., equation (19). This comparison illustrates that for $v < c$ the drag is produced by vortex shedding, whereas for $v > c$, an increasingly significant contribution arises from sound waves. For $v > c$, the reflected matter waves create a standing wave pattern in front of the object as shown in Fig. 4.

Using similar arguments for energy loss of the flow. Consider now the condensate at rest and the obstacle moving with speed v . A drag force leads to energy transfer to the condensate. The energy transfer rate is given by

$$\frac{dE}{dt} = \mathbf{F}_{\text{drag}} \cdot \mathbf{v}. \quad (20)$$

Eqs. (19) and (20) make the important link between vortex shedding, drag and energy dissipation.

5. Motion in a trapped condensate

The inhomogeneous density profile and finite size of trapped condensates means that a steady, uniform flow is difficult to achieve. The MIT experiment [5] partially overcame

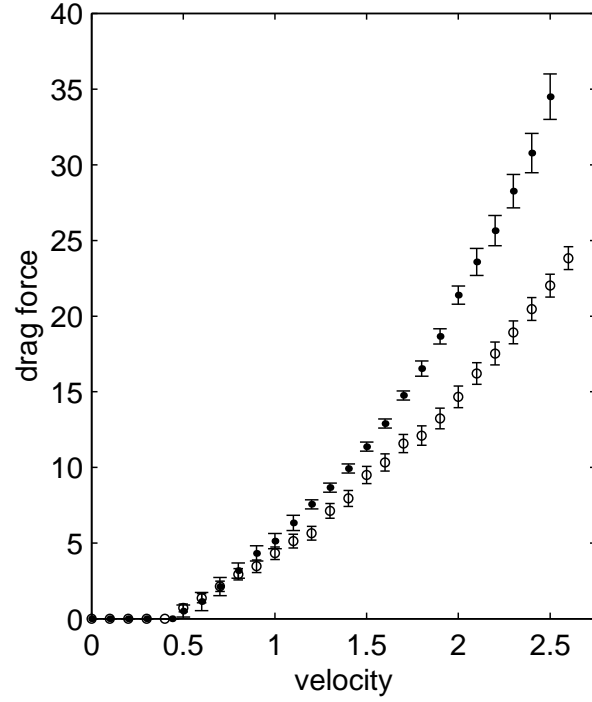


Figure 3. The time-averaged drag force as a function of velocity for an impenetrable cylinder with radius $R = 3$. The open circles indicate the contribution due to vortex shedding. The error bars indicate the residual fluctuations in the time-averaged drag.

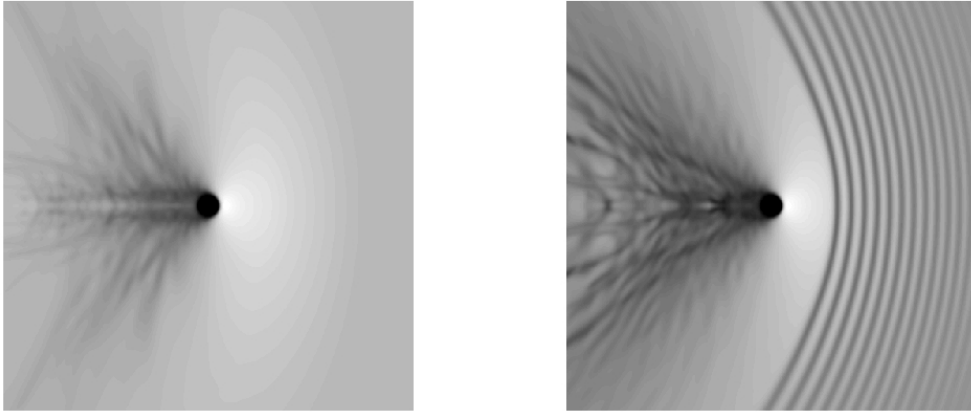


Figure 4. Time-averaged density for homogeneous flow past a cylinder (radius $R = 3$) for flow speeds of 0.9 (left) and 1.2 (right) times the speed of sound. For supersonic flow a standing wave pattern appears in front of the object.

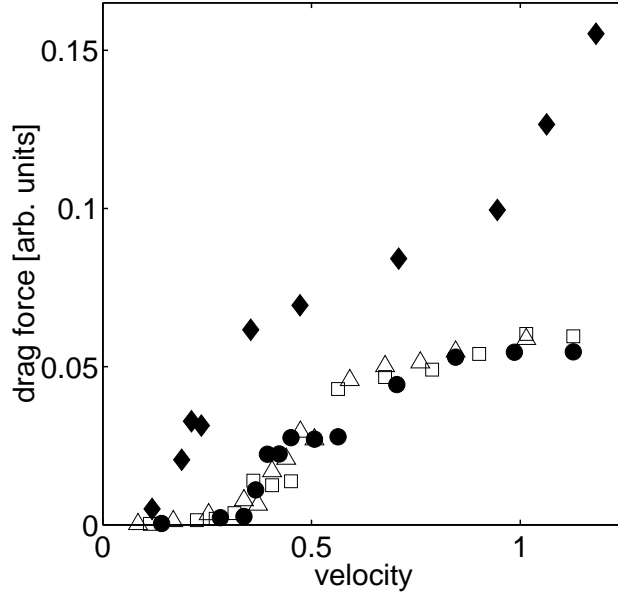


Figure 5. The drag force as a function of velocity for a laser beam moving back and forth through an inhomogeneous condensate. The triangles, squares and circles corresponds to 2D simulations with different oscillation amplitudes. The diamonds correspond to a 3D simulation. In 2D the force is effectively that on a disk of depth 2.5 healing lengths, whereas in 3D the condensate depth corresponds to a Thomas Fermi radius of 6.4 healing lengths.

this problem by sweeping the object back and forth at constant velocity within the central region of the condensate, where the density is approximately uniform. In this case, the object moves through its own low-density wake, and consequently the drag law is different from the uniform flow case discussed above.

In Fig. 5 we show the time-averaged drag on a laser beam oscillating in a trapped condensate. Important differences with the uniform flow case (Fig. 3) arise at high and low velocities. The drag force tends to saturate at higher velocities as the object expels fluid from the region of oscillation and the pressure drops.

Fig. 5 also highlights important differences between two and three dimensions. Two dimensions corresponds to the limit of a ‘cylindrical’ condensate, where the density is uniform parallel to the axis of the object (which we define as the z -axis). However, in realistic three-dimensional situations the density is inhomogeneous along z , leading to a variation of the speed of sound which vanishes at the condensate edge. This results in a lower critical velocity in 3D than in 2D, as apparent in Fig. 5. In the Thomas-Fermi limit ($ga^3 \gg 1$) the density profile is parabolic, therefore the average sound speed along z is a factor of $\pi/4$ smaller than that at the centre. The remaining reduction arises from the fact that even at very low velocities, vortices tend to be formed where the laser beam intersects the lower density fluid at the condensate edge. The lower densities arising in 3D also leads to enhanced sound emission, and hence enhanced drag at high velocities.

Below the critical velocity, dissipation due to sound emission occurs at the motion extrema, where the object accelerates. This is illustrated in Fig. 6 which shows a cross

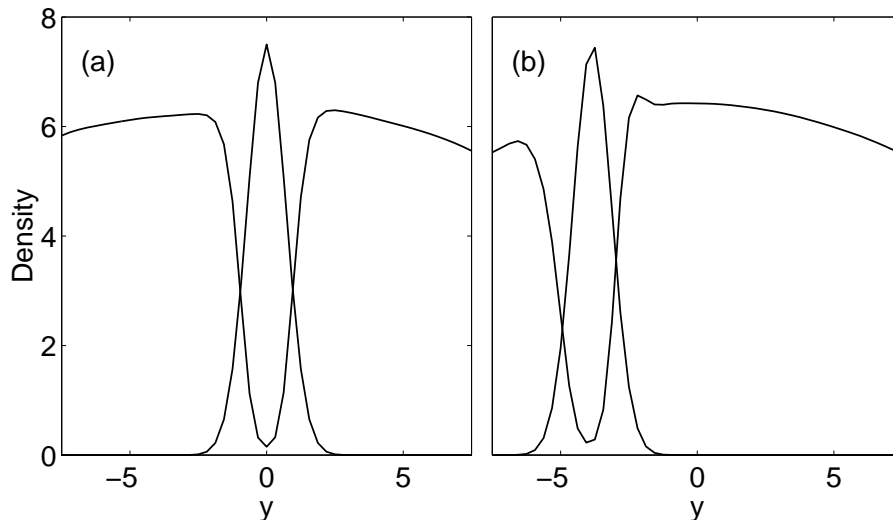


Figure 6. Cross-section through a 2D trapped condensate pierced by a Gaussian laser beam ‘object’ (also shown). (a) If the object is moving at constant velocity, the fluid density is symmetric around the object, and the net drag is zero. (b) If the object accelerates the fluid cannot adjust sufficiently rapidly, leading to an asymmetric fluid distribution around the object which produces drag. The fluid relaxes by the emission of a sound wave. The x -axis is in units of the laser beam waist, while for the y -axis the units are arbitrary.

section through a 2D condensate cut by a moving laser beam. For constant motion, Fig. 6(a), the fluid is distributed symmetrically around the object and the drag, which is given by an overlap integral between the condensate density and the gradient of the object potential (i.e., the second term in equation (10)) is zero. When the object accelerates, Fig. 6(b), the fluid fails to respond rapidly enough to the abrupt change in velocity, and the asymmetry in the overlap between the fluid and the objects leads to a resistance force analogous to dynamic buoyancy. The system relaxes to the uniform flow case by the emission of a sound wave. This corresponds to the ‘phonon heating’ process discussed in [8].

Although the NLSE model describes dissipation in the sense of energy transfer between a moving object and the fluid, it does not say anything about how that energy (mostly stored within the vortex core) may be subsequently converted into heat. A complete description should include coupling of the condensate to a thermal cloud, and would describe the damping of phonon and vortex modes. Recent work on the non-equilibrium dynamics of the condensate and non-condensate predict a depletion of the condensate fraction [16] as observed in the MIT experiment.

6. Summary

The motion of an object through a dilute Bose-Einstein condensate provides an ideal system to study the fundamental problem of the onset of dissipation in superfluids. In this paper we have explained the role of vortex shedding and sound emission in energy transfer between the object and the condensate. No energy transfer is observed under

the condition of uniform, steady flow at speeds below a critical velocity. However, if the object accelerates there is a small dissipative effect, even below the critical velocity, due to sound emission. The critical velocity is reached when the local velocity in the bulk of the fluid (i.e., where the quantum pressure is zero) exceeds the speed of sound. Above the critical velocity vortices are emitted leading to a drag force and energy transfer to the fluid. Vortex shedding dominates the energy transfer for intermediate velocities, while sound emission becomes increasingly important for supersonic motion. We highlight some important differences between homogeneous and inhomogeneous trapped condensates. In particular, that the critical velocity is substantially reduced when the object intersects regions of lower density at the condensate edge and that the trap inhomogeneity gives rise to an additional term in the drag force analogous to a buoyancy.

Acknowledgments

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